We will reconstruct the system possessing a set of spectral data of the form

$$\Lambda^* = \text{diag}\ (1.0,\ 3.5,\ 8.2,\ 11.5,\ 20.0)$$

and

$$Y^* = \begin{bmatrix} 0.40 & -0.56 & 0.50 & 0.30 & 0.37 \\ -0.12 & 0.37 & 0.76 & 0.40 & 0.40 \\ 0.85 & 0.42 & -0.30 & 0.70 & -0.15 \\ 0.25 & -0.10 & 0.45 & -0.20 & 1.21 \\ 0.50 & 1.00 & -0.98 & 0.57 & -0.66 \end{bmatrix}$$

Proceeding thus, one finds

$$M = \begin{bmatrix} 4.78 & 1.15 & 3.21 & 0.22 & 1.72 \\ 1.15 & 3.24 & 2.17 & 5.81 & 5.19 \\ 3.21 & 2.17 & 2.92 & 4.23 & 0.17 \\ 0.22 & 5.81 & 4.23 & 0.83 & 8.31 \\ 1.72 & 5.19 & 0.17 & 8.31 & 1.86 \end{bmatrix}$$

$$K = 10^3 \begin{bmatrix} 1.14 & -1.78 & 1.91 & -0.89 & 3.28 \\ -1.78 & 4.31 & -2.19 & 5.42 & -1.21 \\ 1.91 & -2.49 & 0.87 & -2.71 & 2.09 \\ -0.89 & 5.42 & -2.71 & 3.62 & -4.77 \\ 3.28 & -1.21 & 2.09 & -4.77 & 2.72 \end{bmatrix}$$

One should emphasize that, generally, the effectiveness of the procedure presented here does not depend on the order of the system. Of course, the number of mathematical operations increases when the order of the system increases, but the algorithm does not depend on the number of degrees of freedom.

Conclusions

A method has been proposed for determining modified parameters in symmetric undamped vibratory systems. The method can be applied to a large class of systems with distinct eigenvalues.

The algorithm demonstrated in this Note shows how to change system parameters in order to obtain predetermined spectral data. It is especially useful in practice when we must repeat many times the computations for the same system, but for a different set of eigenvalues and eigenvectors. For the given modified system, after computations of the inverse sensitivities, the increments of the parameters ΔM and ΔK are determined very fast by way of a simple number of matrix multiplications [see Eqs. (20) and (21)]. The method is useful in modifications of complex structures with a large number of degrees of freedom because it is much faster than classical calculations of parameter changes computed on the basis of Eqs. (11) and (12).

One should emphasize that the algorithm presented here allows comparison of the increments ΔM and ΔK produced by the changes of $\Delta\Lambda$ and ΔY, taking into account shares of the individual increments $\Delta\lambda_i$ and Δy_{ij}.

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Error Estimators for Eigenvalues Computed from Discretized Models of Vibrating Structures

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Introduction

NUMERICAL analysis of a solid, plate, or shell structure requires discretization, e.g., by finite elements. Realizing that discretization is accompanied by modeling error, the analyst seeks a level of mesh refinement that is adequate but not excessive. An error estimator for time-independent stress analysis, based on the discrepancy between smoothed and unsmoothed stress fields, has been suggested.[1] The present Note suggests two analogous estimators for error in computed natural frequencies. Both are postprocessing schemes that use the Rayleigh quotient.

In published work related to the present study, the analytical tools are usually called perturbation theory or design sensitivity analysis (e.g., Refs. 2 and 3). There the goal is to predict the change in eigenvalue associated with an actual change in structure stiffness, mass, or support conditions. In the present Note, we make use of the change in eigenvalue associated with changing the representation of stiffness and/or mass, without any changes in the actual structure or in the number of elements used. The motivation is that two different but legitimate representations of stiffness and/or mass will yield two different frequencies for a given mode; hence, the discrepancy between the two frequencies can serve as a measure of computational error for that mode. The second eigenvalue could be obtained by complete resolution of the eigenproblem, but computational economy is greatly enhanced by obtaining it from the Rayleigh quotient instead.

Method 1

For undamped free vibration, the eigenvalue problem is

$$([K_o] - \omega_o^2\,[M_o])\ \{D_o\} = \{0\} \tag{1}$$

where ω_o represents natural frequency, $\{D_o\}$ is a vector of amplitudes of degrees of freedom, and $[K_o]$ and $[M_o]$ are, respectively, the structure stiffness and mass matrices. Subscript o indicates "original," i.e., prior to any modifications of the finite element modeling. Equation (1) is now solved by any convenient eigensolver to obtain natural frequency ω_{oi} and the associated eigenvector $\{D_o\}_i$ for each mode i of interest. Thus far the calculations are standard.

If $[K_o]$, $[M_o]$, and $\{D_o\}_i$ were now entered into the Rayleigh quotient, we would, of course, obtain the known result ω_{oi}^2. Instead, without changing the mesh layout or the number of degrees of freedom, let us evaluate the Rayleigh quotient using an alternative mass matrix $[M_R]$ in place of $[M_o]$, so as to obtain a second natural frequency ω_{Ri}.

$$\omega_{Ri}^2 = \frac{\{D_o\}_i^T\,[K_o]\,\{D_o\}_i}{\{D_o\}_i^T\,[M_R]\,\{D_o\}_i} \tag{2}$$

For example, $[M_o]$ might be the lumped formulation (i.e., diagonal, as from particle masses) and $[M_R]$ the consistent formulation (i.e., derived using the same shape functions used to derive the stiffness matrix). It is well known that frequencies usually converge from below when $[M]$ is lumped and

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converge from above when $[M]$ is consistent. In the present context, Eq. (1) yields ω_{oi} that are usually too low if $[M_o]$ is lumped, and Eq. (2) yield ω_{Ri} that are usually too high if $[M_R]$ is consistent. A significant difference between ω_{oi} and ω_{Ri} suggests that the error in ω_{oi} is significant and that the mesh should be refined. Ideally, ω_{oi} and ω_{Ri} bracket the exact ω_i and have equal magnitudes of error. Accordingly, in subsequent numerical examples, we estimated the percentage error in ω_{oi} as

$$e_i = \frac{\omega_{oi} - \omega_{Ri}}{\omega_{oi} + \omega_{Ri}} 100\% \tag{3}$$

It should not be difficult to provide matrices $[M_o]$ and $[M_R]$ such that ω_{oi} and ω_{Ri} tend to straddle the correct ω_i. There are various formulations, each valid in the sense of converging correctly with mesh refinement, but otherwise having differing properties.[4] These matrices or linear combinations of them will yield suitable $[M_o]$ and $[M_R]$. In the present Note, for lack of space, only one such combination is considered [see Eq. (7)].

Method 2

Method 2 parallels method 1 up to Eq. (2). At this point we observe that $[K_o]\{D_o\}_i$ is a vector of nodal loads, $\{F_o\}_i$ An alternative calculation, discussed below, yields modified nodal loads $\{F_R\}_i$, and the Rayleigh quotient becomes

$$\omega_{Ri}^2 = \frac{\{D_o\}_i^T \{F_R\}_i}{\{D_o\}_i^T [M_R] \{D_o\}_i} \tag{4}$$

in which we may use either $[M_R] = [M_o]$ or choose an alternative for $[M_R]$. As with method 1, a significant difference between ω_{oi} and ω_{Ri} suggests that error is significant and that the mesh should be refined.

Nodal loads $\{F_R\}_i$ are computed by assembly of element contributions $\{f_R\}_i$. These loads are computed in the same fashion as loads produced by initial strain[4]:

$$\{f_R\}_i = \int_{V_e} [B]^T [E] \{\bar{\epsilon}\}_i \, dV \tag{5}$$

where $[B]$ is the element strain-displacement matrix, $[E]$ contains the material stiffnesses, V_e is the element volume, and $\{\bar{\epsilon}\}_i$ is the strain field in the element associated with displacements $\{D_o\}_i$. If $\{\bar{\epsilon}\}_i$ is calculated from element nodal degrees of freedom (DOF) in standard fashion, then strains are discontinuous across interelement boundaries, and we obtain $\omega_{Ri} = \omega_{oi}$. An alternative ω_{Ri} results if $\{\bar{\epsilon}\}_i$ is taken instead from a smoothed stress field. Smoothing involves the computation of nodal average strains, followed by shape-function interpolation over the element from these average values to obtain the element field $\{\bar{\epsilon}\}_i$. The particular smoothing scheme adopted here is called Type 2 by Huang and Cook[5]. Numerical examples show that ω_{Ri} is usually accurate when $[M_R]$ is the consistent mass matrix. Accordingly, we estimate the percentage error in ω_{oi} as

$$e_i = \frac{\omega_{oi} - \omega_{Ri}}{\omega_{Ri}} 100\% \tag{6}$$

Numerical Examples

The test cases are as follows: 1) axial vibration of a uniform bar fixed at one end and modeled by eight elements (8 DOF); 2) lateral vibration of a uniform cantilever beam modeled by four elements (8 DOF); and 3) in-plane vibration of a square membrane fixed along one side and modeled by 4×4 mesh of four-node quadrilaterals (40 DOF). In each quadrilateral, shear stress was taken as constant over the element in the evaluation of $\{f_{Ri}\}$ (but not in the generation of element matrices). A uniform mesh was used in each test case. Mass formulations considered are the lumped form, which is diagonal and has nonzero terms corresponding to translational degrees of freedom only, and the consistent form, which is not diagonal. Also, for the beam problem only, we consider the linear combination

$$[M] = 0.06\,[M_{\text{lumped}}] + 0.94\,[M_{\text{consistent}}] \tag{7}$$

The motivation for this choice is to obtain ω_{oi} and ω_{Ri} values whose errors are comparable in magnitude but of opposite sign. This does not happen if the lumped $[M]$ is used for the beam problem; then, frequencies are seriously underestimated.

Numerical results for method 1 appear in Table 1. Accuracy is satisfactory if $[M_o]$ and $[M_R]$ produce errors of about the

Table 1 Results from method 1

Mode i	ω_{Ri}/ω_{oi}	e_i, % [Eq. (3)]	Actual errors ω_{oi}, %	Actual errors ω_{Ri}, %
	Bar: $[M_o]$ lumped, $[M_R]$ consistent			
1	1.003	−0.2	−0.2	0.2
2	1.029	−1.4	−1.4	1.5
3	1.084	−4.0	−4.0	4.0
4	1.169	−7.8	−7.7	7.9
	Beam: $[M_o]$ lumped, $[M_R]$ consistent			
1	1.029	−1.4	−2.8	0.0
2	1.095	−4.5	−8.8	−0.2
3	1.135	−6.3	−13.8	−2.1
4	1.262	−11.6	−23.3	−3.2
	Beam: $[M_o]$ from Eq. (7), $[M_R]$ consistent			
1	1.002	−0.1	−0.2	0.0
2	1.006	−0.3	−0.5	0.1
3	1.012	−0.6	−0.4	0.8
4	1.027	−1.3	−1.3	1.4
	Membrane: $[M_o]$ lumped, $[M_R]$ consistent			
1	1.023	−1.1	0.8	3.2
2	1.015	−0.7	−0.4	1.1
3	1.092	−4.4	−3.2	5.8
4	1.246	−10.9	−11.3	10.5
5	1.128	−6.0	−2.5	10.0
6	1.118	−5.6	−4.5	6.7

Table 2 Results from method 2

Mode i	ω_{Ri}/ω_{oi}	e_i, % [Eq. (6)]	Actual errors ω_{oi}, %	Actual errors ω_{Ri}, %
	Bar: $[M_o]$ lumped, $[M_R]$ consistent			
1	1.002	−0.2	−0.2	0.0
2	1.017	−1.6	−1.4	0.2
3	1.048	−4.6	−4.0	0.6
4	1.098	−9.0	−7.7	1.4
	Bar: $[M_o]$ and $[M_R]$ both consistent			
1	0.999	0.1	0.2	0.0
2	0.988	1.3	1.5	0.2
3	0.967	3.4	4.0	0.6
4	0.939	6.4	7.9	1.4
	Membrane: $[M_o]$ lumped, $[M_R]$ consistent			
1	0.998	0.2	0.8	0.6
2	1.004	−0.4	−0.4	0.1
3	1.046	−4.4	−3.2	1.3
4	1.157	−13.6	−11.3	2.7
5	1.027	−2.6	−2.5	0.1
6	1.056	−5.3	−4.5	0.8
	Membrane: $[M_o]$ and $[M_R]$ both consistent			
1	0.976	2.4	3.1	0.7
2	0.990	1.0	1.1	0.1
3	0.960	4.1	5.7	1.5
4	0.948	5.5	8.3	2.7
5	0.924	8.2	9.4	1.1
6	0.945	5.9	7.0	1.0

same magnitude but opposite sign. It then makes little difference whether $[M_o]$ is lumped and $[M_R]$ is consistent or vice versa. A usually conservative interpretation is that the average value $(\omega_{oi} + \omega_{Ri})/2$ is in error by e_i, at most.

Numerical results for method 2 appear in Table 2. The beam problem is omitted because its moment field is already inter-element-continuous and would therefore yield error estimates of zero. Accuracy is satisfactory if ω_{Ri} is reasonably accurate, as usually happens if $[M_R]$ is the consistent mass matrix.

The foregoing results are encouraging, but substantiation from larger and more complicated problems is desirable.

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Generalized Kelvin Function Solutions for a Class of Vibrating Circular-Plate Problems

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Introduction

THE study of the vibration of a circular plate finds many important applications. In this Note, we consider a class of vibration problems of a circular plate subjected to a periodic excitation force and a viscous damping force. Two examples are the forced vibrations of 1) the injector plate due to pressure oscillation in the combustion chamber of a liquid rocket engine such as the Space Shuttle main engine,[1] and 2) the circular lid of an underwater container subjected to water motion. The governing equation for the transverse deflection of the plate is[2]

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} + c\frac{\partial w}{\partial t} = p \tag{1}$$

where

$$D = Eh^3/[12/(1-\nu^2)] \tag{2}$$

is the flexural rigidity; E is the Young's modulus; h is the plate thickness; ν is the Poisson ratio; $\nabla^4 = \nabla^2\nabla^2$ is the biharmonic differential operator; and for the circular plate, it is expressed in polar notation:

$$\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial^2\theta}\right)^2 \tag{3}$$

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In Eq. (1), w is the transverse deflection of a typical point on the plate; ρ is the mass per unit volume; t is the time; c is the viscous damping coefficient; and p is the transverse excitation force.

With zero values of c and p and homogenous boundary conditions, Eq. (1) constitutes an eigenvalue problem for the vibration of the plate. The classical method for structural vibrations is by eigenfunction superposition,[2] whereas the present paper solves the structural vibration as a boundary value problem in a straightforward manner. It yields a closed-form (instead of infinite series) exact solution, which is expressed in terms of the first and second kinds of Bessel functions with complex arguments. No formulas are available for these types of functions. This Note introduces a new set of modified Bessel functions for the complex variables, called functions H and Q. By using this new set of functions, the displacement, velocity, internal force, or bending moments of the elastic vibration of the circular plate can be readily calculated.

Formulation and Solution

Since we are considering a periodic excitation force, it is possible to expand both the force p and displacement function w in a complex form of the Fourier series:

$$p = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_{mn}(r)\, e^{i(m\theta + \omega n t)}$$

$$w = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} w_{mn}(r) e^{i(m\theta + \omega n t)} \tag{4}$$

where m is an integer representing the mth transverse mode of plate vibration, θ is the angular coordinate measured in the circumferential direction of the plate, and ω is the first harmonic frequency of the periodic force.

Now Eq. (1) becomes

$$D\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)^2 w_{mn} - \rho h(\omega n)^2 w_{mn} + i\, c\omega n\, w_{mn} = p_{mn} \tag{5}$$

First, let us find the complementary solution of Eq. (5) in the form

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)^2 w_{nnc} - k_n^4 w_{mnc} = 0 \tag{6}$$

where

$$k_n = \left\{\rho h(\omega n)^2 \frac{[1 - i\, c/(\rho h\omega n)]}{D}\right\}^{1/4} = s_n \exp\left(\frac{i\psi}{4}\right) \tag{7a}$$

$$s_n = \left\{\rho h(\omega n)^2 \frac{[1 + (\mu/n)^2]^{1/2}}{D}\right\}^{1/4} \tag{7b}$$

$$\mu = c/(\rho h\omega), \qquad \psi = \arctan(-\mu/n) \tag{7c}$$

Equation (6) can be further separated into

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right) w_{mnc} + k_n^2 w_{mnc} = 0 \tag{8a}$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right) w_{mnc} - k_n^2 w_{mnc} = 0 \tag{8b}$$

The solution to Eq. (8a) is in terms of the mth-order Bessel functions of the first and second kinds, $J_m(k_n r)$ and $Y_m(k_n r)$, and that of Eq. (8b) is in terms of the mth-order modified Bessel functions of the first and second kinds, $I_m(k_n r)$ and $K_m(k_n r)$. The solution to Eq. (5) is therefore

$$w_{mn} = C_1 J_m(k_n r) + C_2 Y_m(k_n r) + C_3 I_m(k_n r) + C_4 K_m(k_n r) + w_{mnp} \tag{9}$$